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A Comparative Analysis of Mathematical Linear Programming Problem

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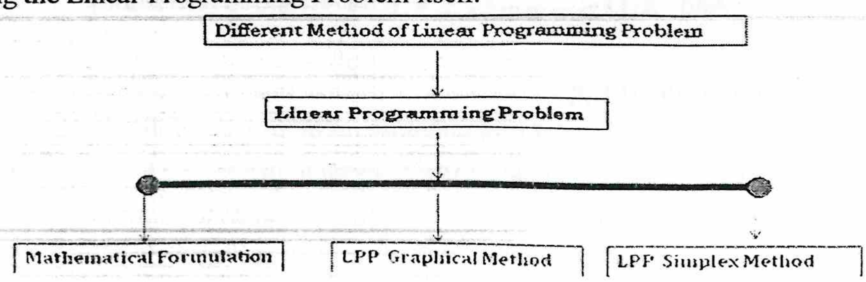
Abstract:-

In this paper Comparative Analysis of Linear Programming problems arise in different method of solving LPP such as Mathematical Formulation, Linear Programming Problem Graphical Method and LPP in Simplex Method. The Simplex Method is matrix based method used for solving linear programming problems with any number of variables. The Linear Programming Problem is a some special case of Mathematical Formulation. Depending on the objective we want to optimize Minimize or Maximize, we obtain the typical Linear programming Problem. Also introduce linear programming, one of the most powerful tools in operations research. Thus equipped, we then venture into some of the many applications that can be modeled with linear programming. This subsection will first demonstrate how to plot constraints, and then show how to deal with objective functions, and then put it all together in the graphical solution method. As with the graphical method, the simplex method finds the most attractive corner of the feasible region to solve the LP problem. Also some of the variations and some special cases in Linear Programming Problem and its Comparative Analysis have been discussed in the paper.

Keywords:-*Mathematical linear programming problem, Linear Programming Problem(LPP), Graphical Method of LPP, Simplex Method.*

Introduction

The Linear Programming Problem (LPP) of the Mathematical Programming consists of a linear objective function and a set of linear constraints to be fulfilled. It is simple in structure but powerful in applicability to a wide range of problems in engineering, medicine, agriculture, industry, business, social science, management, defense, administration, communication etc... It has become an important model in modem theoretical and applied mathematics and has proved to be one of the most effective tools in operations research for decision making. The success of it stems from its flexibility in depicting real world situations varying from military operations to behavioral and social sciences. It is a model to get the best out of a given situation. Linear Programming Problems are concerned with the optimal allocation of limited resources to meet certain desired objective. The linear function, which is to be optimized, is called the objective function and the conditions to be satisfied are expressed as simultaneous linear equalities or inequalities referred to as constraints. A solution vector, which satisfies the set of constraints and the given objective, is termed as an optimal solution. The logical analysis and conclusions of all decision making problems are based on the concept of models and model building, which is an abstraction of the reality and may appear to be less complex than the reality itself. The collection of data is the most difficult part of constructing a model. The formulation of a linear programming model involves a detailed study of the system, data collection, identification of decision variables, and construction of the objective function and system constraints. In the course of formulating a model, model builders often tend to include inadvertently or innocently, constraints and variables that may not play a role at all in defining the feasible set. The presence of such constraints / variables is hardly disputed and can play havoc with Linear Programming solution procedures and greatly hamper the solution effort. It is well known that every additional constraint / variable in a Linear Programming Problem demands more computational effort and computer memory. The identification of such embedded non-role play constraints / variables in the model without affecting the character of the system is as difficult as solving the Linear Programming Problem itself.



Linear Programming Problem

Linear programming is a powerful quantitative technique (or operational research technique) designs to solve allocation problem. The term 'linear programming' consists of the two words 'Linear' and 'Programming'. The word 'Linear' is used to describe the relationship between decision variables, which are directly proportional. For example, if doubling (or tripling) the production of a product will exactly double (or triple) the profit and required resources, then it is linear relationship. The word 'programming' means planning of activities in a manner that achieves some 'optimal' result with available resources. A program is 'optimal' if it maximizes or minimizes some measure or criterion of effectiveness such as profit, contribution (i.e. sales-variable cost), sales, and cost. Thus, 'Linear Programming' indicates the planning of decision variables, which are directly proportional, to achieve the 'optimal' result considering the limitations within which the problem is to be solved.

Decision Variables: - The decision variables refer to the economic or physical quantities, which are competing with one another for sharing the given limited resources. The relationship among these variables must be linear under linear programming. The numerical values of decision variables indicate the solution of the linear programming problem.

Objective Function: - The objective function of a linear programming problem is a linear function of the decision variable expressing the objective of the decision maker. For example, maximization of profit or contribution, minimization of cost/time.

Constraints: - The constraints indicate limitations on the resources, which are to be allocated among various decision variables. These resources may be production capacity, manpower, time, space or machinery. These must be capable of being expressed as linear equation (i.e. =) on inequalities (i.e. > or <; type) in terms of decision variables. Thus, constraints of a linear programming problem are linear equalities or inequalities arising out of practical limitations.

Non-negativity Restriction: - Non-negativity restriction indicates that all decision variables must take on values equal to or greater than zero.

Divisibility: - Divisibility means that the numerical values of the decision variables are continuous and not limited to integers. In other words, fractional values of the decision variables must be permissible in obtaining optimal solution.

Certainty: - In all LP models it is assumed that, all the model parameters such as availability of resources, profit contribution of a unit of decision variable and consumption of resources by a unit of decision variable must be known and constant.

The General Lpp And Computational Procedures

The general linear programming problem as under: Subject Maximize (Minimize)

$$Z = C_1X_1 + C_2X_2 + \dots\dots\dots + C_nX_n$$

subject to conditions

$$a_{11}X_1 + a_{12}X_2 + \dots\dots\dots + a_{1n}X_n \leq b_1 \text{(Maximum availability)}$$

$$a_{21}X_1 + a_{22}X_2 + \dots\dots\dots + a_{2n}X_n \geq b_2 \text{ (Minimum availability)}$$

$$a_{31}X_1 + a_{32}X_2 + \dots\dots\dots + a_{3n}X_n = b_3 \quad \text{(Equality)}$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \dots\dots\dots + a_{mn}X_n \leq = \geq b_m$$

And $x_1; x_2; \dots\dots x_n \geq 0$ (Non-Negative restriction)

The Mathematical Formulation Of Lpp

The steps involved in the formation of linear programming problem are as follows:

Step 1: Identify the Decision Variables of interest to the decision maker and express them as $X_1, X_2, X_3 \dots$

Step 2: Ascertain the Objective of the decision maker whether he wants to minimize or to maximize.

Step 3: Ascertain the cost or the profit per unit of each of the decision variables.

Step 4: Ascertain the constraints representing the maximum availability or minimum commitment or equality and represent them as less than or equal to (≤) type inequality or 'equal to' (=) type equality respectively.

Marketing Problem:

The P Q stone company sells stone secured from any of the two adjacent quarries. The stone sold by the company must conform to the following specifications.

Material X equal to 40% and material Y equal to 30%.

Stone from quarry a cost Rs. 15 per tone and has the following properties.

Material X: 4%, Material Y: 2%

Stone from quarry B cost Rs. 20 per tone and has following properties

Material X: 2%, Material Y: 3%

Formulation Mathematical

The data of the problem is summarized below

Material	Quarry		Specifications
	A	B	
X	4	2	40
Y	2	3	30
Cost per tone	15	20	

Max $Z = 15x_1 + 24x_2$ Subject to constraints

$$4x_1 + 2x_2 \le 40 , 2x_1 + 3x_2 \le 30 \text{ and } x_1, x_2 \ge 0$$

Linear Programming Problem In Graphical Method

Graphical method of linear programming is used to solve problems by finding the highest or lowest point of intersection between the objective function line and the feasible region on a graph

Closed Half Plane: - A linear inequality in two variables is known as a half plane. The corresponding equality or the line is known as the boundary of the half plane. The half plane along with its boundary is called a closed half plane. Thus, a closed half plane is a linear inequality in two variables, which include the value of the variable for which equality is attained.

Feasible Solution: - Any non-negative solution which satisfies all the constraints is known as a feasible solution of the problem.

Feasible Region: - The collection of all feasible solutions is known as a feasible region.

Convex Set: - A set (or region) is convex if only if for any two points on the set, the line segment joining those points lies entirely in the set. Thus, the collection of feasible solutions in a linear programming problem forms a convex set. In other words, the feasible region of a linear programming problem is a convex set.

Convex Polygon: - A convex polygon is a convex set formed by the intersection of a finite number of closed half planes.

Extreme Points or Vertexes or Corner Points:- The extreme points of a convex polygon are the points of intersection of the lines bounding the feasible region. The value of the decision variables, which maximize or minimize the objective function, is located on one of the extreme points of the convex polygon. If the maximum or minimum value of a linear function defined over a convex polygon exists, then it must be on one of the extreme points.

Redundant Constraint: - Redundant constraint is a constraint, which does not affect the feasible region. Multiple Solutions Multiple solutions of a linear programming problem are solutions each of which maximize or minimize the objective function. Under graphical method, the existence of multiple solutions is indicated by a situation under which the objective function line coincides with one of the half planes generated by a constraint. In other words, where both the objective function line and one of constraint lines have the same slope.

Unbounded Solution: - An unbounded solution of a linear programming problem is a solution whose objective function is infinite. A linear programming problem is said to have unbounded solution if its solution can be made infinitely large without violating any of the constraints in the problem. Since there is no real applied problem, which has infinite returns, hence an unbounded solution always represents a problem that has been incorrectly formulated. For example, in a maximization problem at least one of the constraints must be an equality' or 'less than or equal to' (≤) type, then there will be no upper limit on the feasible region. Similarly for minimization problem, at least one of constraints must be an 'equality' or 'a greater than or equal to' type (≥) if a solution is to be bounded. Under graphical method, the feasible solution region extends indefinitely.

Infeasible Problem A linear programming problem is said to be infeasible if there is no solution that satisfies all the constraints. It represents a state of inconsistency in the set of constraints.

An Idea of linear programming

If the feasible set of a linear programming problem with two variables is bounded (contained inside some big circle; equivalently, there is no direction in which you can travel indefinitely while staying in the feasible set), then, whether the problem is a minimization or a maximization, there will be an optimum value. Furthermore:

- there will be some corner point of the feasible region that is an optimum
- if there is more than one optimum corner point then there will be exactly two of them, they will be adjacent, and any point in the line between them will also be optimum.

Practical Steps Involved In Solving Lpp By Graphical Method

The practical steps involved in solving linear programming problems by Graphical Method are given below:

Step 1: Consider each inequality constraint as equation.

Step 2: Take one variable (say x) in a given equation equal to zero and find the value of other variable (say y) by solving that equation to get one co-ordinate [say (0, y)] for that equation.

Step 3: Take the second variable (say y) as zero in the said equation and find the value of first variable (say x) to get another co-ordinate [say (x, 0)] for that equation.

Step 4: Plot both the co-ordinates so obtained [i.e., (0, y) and (x, 0)] on the graph and join them by a straight line. This straight line shows that any point on that line satisfies the equality and any point below or above that line shows inequality. Shade the feasible region which may be either convex to the origin in case of less than type of inequality.

Step 5: Repeat Steps 2 to 4 for other constraints.

Step 6: Find the common shaded feasible region and mark the co-ordinates of its comer points.

Step 7: Put the co-ordinates of each of such vertex in the Objective Function. Choose that vertex which achieves the most optimal solution.

Table:-

Vertex No	Co-ordinates of vertices of common Shaded Feasible region	Value of Z
1	(x_1, y_1)	$Z_1 =$
2	(x_2, y_2)	$Z_2 =$
3	(x_3, y_3)	$Z_3 =$
4	(x_4, y_4)	$Z_4 =$

Step: Let x_1, x_2 represent the proportion of stone in tone to be produced from quarries A and B

Objective function: The objective function is to minimize the total cost i. e. Max Z= $15x_1 + 20x_2$

Constraints: The constraints are on the two types material of the stone there are graphical method of solving LPP

$$\text{Max } Z = 15x_1 + 24x_2$$

subject to constraints

$$4x_1 + 2x_2 \leq 40, 2x_1 + 3x_2 \leq 30 \text{ and } x_1, x_2 \geq 0$$

Solution: Let us convert each both of the constraints from '≤ 'type to' =' type

$4x_1 + 2x_2 = 40, 2x_1 + 3x_2 = 30$ Now we draw corresponding lines for each equation as follows.

The first constraints $4x_1 + 2x_2 = 40$

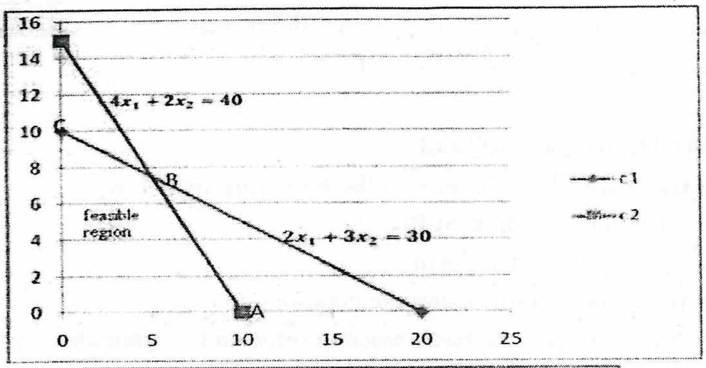
When $x_1 = 0$ *we get* $4 \times 0 + 2x_2 = 40$ *implies* $x_2 = 20$

Similarly $x_2 = 0$ we get $2x_1 + 2 \times 0 = 30$ implies $x_1 = 10$

The second constraints $2x_1 + 3x_2 = 30$

When $x_1 = 0$ we get $2 \times 0 + 3x_2 = 30$ implies $x_2 = 10$

Similarly $x_2 = 0$ we get $2x_1 + 3 \times 0 = 30$ implies $x_1 = 15$



Point	x_1	x_2	$z = 15x_1 + 2x_2$
O	0	0	0
A	10	0	150
B	7	5	205 Maximize
C	0	10	200

Simplex Method

It was developed by G. Danztig in 1947. The simplex method provides an algorithm which is based on the fundamental theorem of linear programming. If the feasible region to any linear programming problem has at least one point and is convex and if the objective function has a maximum (or minimum) value within the feasible region, then the maximum (or minimum) will always occur at a corner point in that region. Criteria for the existence of solutions, $f(x,y) = ax + by$ If the feasible region is bounded, then has a maximum and minimum. If the feasible region is unbounded and $a, b > 0$, then f has a minimum; but not a maximum. The Simplex Method is "a systematic procedure for generating and testing candidate vertex solutions to a linear program." (Gill, Murray, and Wright, p. 337) It begins at an arbitrary corner of the solution set. At each iteration the Simplex Method selects the variable that will produce the largest change towards the minimum (or maximum) solution. That variable replaces one of its compatriots that is most severely restricting it, thus moving the Simplex Method to a different corner of the solution set and closer to the final solution. In addition, the Simplex Method can determine if no solution actually exists.

The Simplex Method solves a linear program of the form described in Figure. Here, the coefficients C_j represent the respective weights, or costs, of the variables X_j. The minimized statement is similarly called the cost of the solution. The coefficients of the system of equations are represented by, a_{ij} and any constant values in the system of equations are combined on the right-hand side of the inequality in the variables b_i. Combined, these statements represent a linear program, to which we seek a solution of minimum cost.

Optimal Solution	Type of Problem	Optimum Solution
a)	In case of maximization problem	Maximum value of Z is the optimal solution
b)	In case of minimization problem	Minimum value of Z is the optimal solution

The following important definitions are necessary for understanding and developing the theory those fellows.

Objective Function: - The function that is either being minimized or maximized. For example, it may represent the cost that you are trying to minimize.

Optimal Solution:- A vector x, which is both feasible (satisfying the constraints) and optimal (obtaining the largest or smallest objective value).

Constraints:- A set of equalities and inequalities that the feasible solution must satisfy.

Feasible Solution:- A solution vector, x, which satisfies the constraints.

Basic Solution: - It is a solution obtained by setting any 'n' variable equal to zero and solving remaining 'm' variables such 'm' variables are called Basic variables and 'n' variables are called Non-basic variables.

Slack Variable A variable added to the problem to eliminate less-than constraints.

Surplus Variable A variable added to the problem to eliminate greater-than constraints.

Artificial Variable A variable added to a linear program in phase 1 to aid finding a feasible solution.

Unbounded Solution For some linear programs it is possible to make the objective arbitrarily small (without bound). Such an LP is said to have an unbounded solution.

Computational Procedure Of Simplex Method

The optimal solution to a General LPP is obtained in the following major steps:

Step 1 - Write the given GLPP in the form of SLPP

Step 2 - Present the constraints in the matrix form

Step 3 - Construct the starting simplex table using the notations.

Step 4 - Calculation of Z and Δj and test the basic feasible solution for optimality by the rules given. Z= $C_B X_B$. Procedure to test the basic feasible solution for optimality by the rules given.

- If all $\Delta_j \geq 0$, the solution under the test will be optimal. Alternate optimal solution will exist if any non-basic Δ_j is also zero.
- If at least one Δ_j is negative, the solution is not optimal and then proceeds to improve the solution in the next step.
- If corresponding to any negative Δ_j all elements of the column $X_□$ are negative or zero, then the solution under test will be unbounded. In this problem it is observed that Δ_1 and Δ_2 are negative. Hence proceed to improve this solution

Step 5 – To improve the basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined.

Incoming vector: - The incoming vector $X_□$ is always selected corresponding to the most negative value of $\Delta_□$. It is indicated by (↑).

Outgoing vector: - The outgoing vector is selected corresponding to the least positive value of minimum ratio. It is indicated by (→).

Step 6–Mark the key element or pivot element by '1'the element at the intersection of outgoing vector and incoming vector is the pivot element.

- If the number in the marked position is other than unity, divide all the elements of that row by the key element.
- Then subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeroes in the remaining position of the column $X_□$.

Step 6– Now repeat step 4 through step 6 unit an optimal solution is obtained.

Examples: - Using simplex method to solve the following LPP

Max: $Z= 15□_1 + 20□_2 + 0□_1 + 0□_2$Subject to constraints

$4□_1 + 2□_2 + □_1 = 40,\ 2□_1 + 3□_2 + □_2 = 30$

Preset the constraints in the matrix form

$$\begin{bmatrix} 4 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} □_1 \\ □_2 \\ □_1 \\ □_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \end{bmatrix}$$

Simplex table

First Iteration Table

Basic Variables		$C_□$	15	20	0	0	min ratio
	$C_□$	$X_□$	$□_1$	$□_2$	$□_1$	$□_2$	

$\square_1$	0	40	4	2	1	0	40/2 = 20
$\square_2$	0	30	2	[3]	0	1	30/3 = 10
	$Z = \square_{\square}$	$\square_{\square=0}$	-15	-20	0	0	

$$Z = \square_{\square}\,\square_{\square} = [0 \quad 0]\begin{bmatrix}40\\30\end{bmatrix} = 0$$

$\Delta_{\square} = \square_{\square} - \square_{\square} = \square_{\square}\square_{\square} - \square_{\square}$

Second Iteration Table

Basic Variables		$\square_{\square}$	15	20	0	0	min ratio
	$\square_{\square}$	$\square_{\square}$	$\square_1$	$\square_2$	$\square_1$	$\square_2$	
$\square_1$	0	20	-5/3	0	1	-7/3	
$\square_2$	20	10	3	1	0	1/3	
	$\square_{\square}\ \square_{\square}$	200	45	0	0	0	

$$Z = \square_{\square}\,\square_{\square} = [0 \quad 20]\begin{bmatrix}20\\10\end{bmatrix} = 20*10 = 200$$

$\Delta_{\square} = \square_{\square} - \square_{\square} = \square_{\square}\square_{\square} - \square_{\square}$

Therefore solution is max Z=200

Conclusion

We study Linear Programming problems arise in different method of solving LPP such as Mathematical Formulation, Linear Programming Problem Graphical Method and LPP in Simplex Method. Also in this paper presents a insight into understanding, analyzing a linear programming problem and obtaining a optimum solution (maximization or minimization) by simplex method. The procedure and algorithm of simplex method with examples are discussed in detail to understand the LPP more precisely and effectively. Give a Marketing problem in a data center, and an application of Linear Programming problems are a well studied topic in combinatorial optimization. These problems find numerous applications in production planning, engineering, medicine, agriculture, industry, business, social science, management etc.

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